Journal of Positive School Psychology
2023, Vol. 7, Issue 2
Pp 134-150

Modified transmuted family of distributions: an application to the bladder cancer data

Noor Saeed Khan
Alamgir
Department of Statistics, University of Peshawar, KP, Pakistan

Abstract

Distribution theory gives a unified approach to dealing with model description and data interpretation problems. In many applied sciences such as medicine, engineering and finance, modeling and analyzing lifetime data are critical. This paper contributes a new family of distributions. This new family of distributions is referred as Modified Transmuted (MT) family of distributions. The characteristics of this new family are studied through analytical, graphical and numerical approach. A special case, where baseline distribution of family is Weibull, has been considered in details. The new version is called Modified Transmuted Weibull (MTW) distribution. This new distribution presents both increasing and decreasing failure rate functions, which is of great utility in reliability and survival analysis. Some properties of the proposed distribution, including estimation of parameters, hazard function, the order statistics, mean residual, quantile function, mode and expression for the moments are obtained. A simulation study has been carried out. Furthermore, a real data set related to remission time of bladder cancer patients has been analyzed to show how they consider model work in practice. is used to illustrate the importance of the proposed distribution. From the results and findings, it has been observed that the new distribution work better for lifetime data.

Keywords: Modified Transmuted family, order statistics, mean residual, mode, quantile function, Skewness, Weibull distribution.

Article History:
Received: 20th February, 2023
Accepted: 25nd March, 2023
Published: 29th April, 2023

1. Introduction

In the field of distribution theory, the derivation of new family and new distribution has become popular. Several efforts have been made to introduce new family of distributions so that to extend the already available models. In the recent research, various families of distributions have been developed to introduce new flexible models. Afify et al. (2016) developed the Kumaraswamy transmuted G family of distributions. Corderio and Castro (2011) proposed Kumaraswamy type-1 class of distributions, its cumulative distribution function (CDF) is given by

$$G(x)=1-\left(1-\left(F(x)\right)^{\alpha}\right)^{\beta} \qquad \alpha,\beta>0 \qquad (1)$$

Cordeiro et al. (2013) developed a new method known as exponentiated generalized class of distribution. Gupta and Kundu (2001) proposed exponentiated exponential distribution. Khan and King (2013) worked on transmuted modified Weibull distribution. Lee and Wang (2003) used statistical method for survival data analysis. Pareto distribution was used by Levy & Levy (2003) for examination of prosperity in humanity. Mahadavi and Kundu (2015) produced a new technique for developing distribution(s) and they called it as alpha power transformation (APT) technique.

The CDF of the APT technique is given as

$$G(x)=\begin{cases}\dfrac{\alpha^{F(x)}-1}{\alpha-1} & \alpha>0,\alpha\neq 1\\ F(x) & \alpha=1\end{cases} \qquad (2)$$

A new technique was suggested by Marshal and Olkin (1997). Mudholkar and Srivastava (1993) presented a new technique to incorporate a new parameter to the baseline distributions. Exponentiated Pareto distribution was presented by Nadarajah (2005). The one parameter Pareto distribution was introduced by Philbrick (1985). Smith and Naylor (1987) developed a three parameters Weibull distribution and used it for comparison of maximum likelihood and Bayesian estimators. A new family of distribution called Kumaraswamy type 2 was proposed by Tahir and Nadarajah (2015). The cumulative distribution function (CDF) of this family of distribution is defined by

$$F(x) = 1 - \left(1 - \left(1 - G(x)\right)^{\beta}\right)^{\lambda} \qquad \beta, \lambda > 0 \qquad (3)$$

2. Bladder Cancer Patients Data:

The bladder cancer patient's data used by different authors in different time. Some of the authors who utilized the bladder cancer patient's data for their distribution are presented here. Lemonte and Cordeiro (2011) developed extended Lomax distribution and they applied it to bladder cancer patients. Using bladder cancer patient's data, Khan et al. (2014) produced characterization of the transmuted inverse Weibull distribution. Kilany (2016) worked on Weighted Lomax distribution. Yadav et al. (2016) developed on hybrid censored inverse Lomax distribution its application to the survival data. The half-Logistic Lomax distribution for Lifetime Modeling was presented by Anwar and Zahoor (2018). Tomy (2018) worked on a retrospective study on Lindely distribution. Reyes et al. suggested modified slash Lindely distribution. Babtain et al. (2020) developed Sine Topp-Leone-G family of distribution. Merovci (2013) introduced transmuted Lindely distribution.

In this paper, a new family of distribution is presented referred Modified Transmuted Family. The special case of this family is obtained by incorporating the Weibull distribution called Modified Transmuted Weibull (MTW) distribution. Bladder cancer patient's data are used to check the efficiency of the suggested model.

3. Modified Transmuted family of distributions

This section illustrates a new family of probability distributions called Modified Transmuted Family of distribution. Let x be a continuous random variable then cumulative distribution function (CDF) of the Modified Transmuted family of distribution is defined as

$$G(x) = \frac{F(x)\left(\alpha - F(x)\right)}{\alpha - 1} \qquad \alpha \ge 2, x \ge 0 \qquad (4)$$

From the above equation, we can clearly show that G(x) is the weighted function of F(x), where the weight is $w(x) = \frac{\alpha - F(x)}{\alpha - 1}$, so equation (4) can also express as $G(x) = F(x)w(x)$

Weighted distributions play an important role in reliability, survival analysis and biomedicine.

The corresponding probability density function (PDF) of the new family of distribution is given by

$$g(x) = \frac{f(x)\left(\alpha - 2F(x)\right)}{\alpha - 1} \qquad \alpha \ge 2, x \ge 0 \qquad (5)$$

Weibull Distribution

The Weibull distribution is one of the most popular and widely used model in life-testing and reliability theory. The cumulative distribution function (CDF) of the random variable X has the Weibull distribution is given by

$$F(x) = 1 - e^{-\lambda x^{\beta}} \qquad \alpha, \beta, x > 0 \qquad (6)$$

The probability distribution function (PDF) of the Weibull distribution is given by

$$f(x) = \lambda \beta x^{\beta - 1} e^{-\lambda x^{\beta}} \qquad \lambda, \beta, x > 0 \qquad (7)$$

Modified Transmuted Weibull (MTW) Distribution

If F(x) is the Weibull cumulative distribution function (CDF), equation (1) yields the NEAPP cumulative distribution function as

$$F(x) = \frac{(1 - e^{-\lambda x^{\beta}})(\alpha - 1 + e^{-\lambda x^{\beta}})}{\alpha - 1} \qquad x, \beta, \lambda > 0, \alpha \geq 2 \qquad (8)$$

$$f(x) = \frac{\beta \lambda x^{\beta - 1} e^{-\lambda x^{\beta}} (\alpha - 2 + 2e^{-\lambda x^{\beta}})}{\alpha - 1} \qquad x, \beta, \lambda > 0, \alpha \geq 2 \qquad (9)$$

The pdf of MTW has been plotted in Figure 1 below.

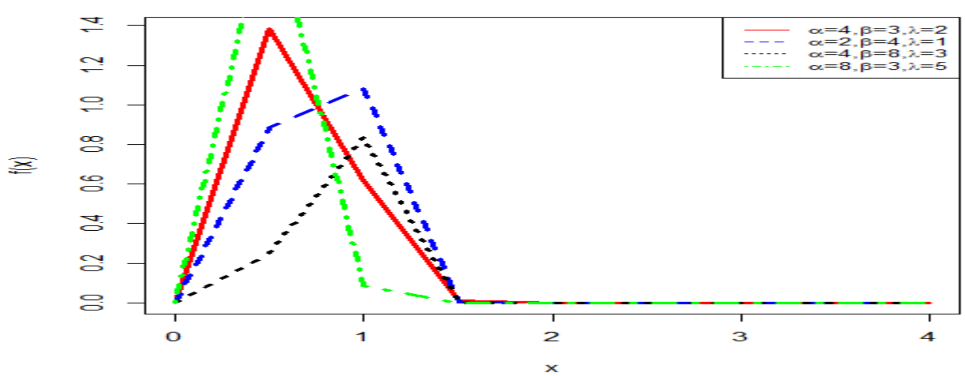

Fig 1: graph of the Probability Density Function of MTW distribution

3. The Survival Function (SF)

As we know that the expression for survival function is written as

$$S(x) = 1 - F(x)$$

Using CDF of modified transmuted Weibull distribution, we obtained

$$S(x) = 1 - \frac{(1 - e^{-\lambda x^{\beta}})(\alpha - 1 + e^{-\lambda x^{\beta}})}{\alpha - 1}$$

After simplification, we get the following expression

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$$S(x)=\frac{\alpha-1-\left((1-e^{-\lambda x^{\beta}})(\alpha-1+e^{-\lambda x^{\beta}})\right)}{\alpha-1} \qquad (10)$$

Figure 2 displays the SF of the MTW distribution for various values of the parameters.

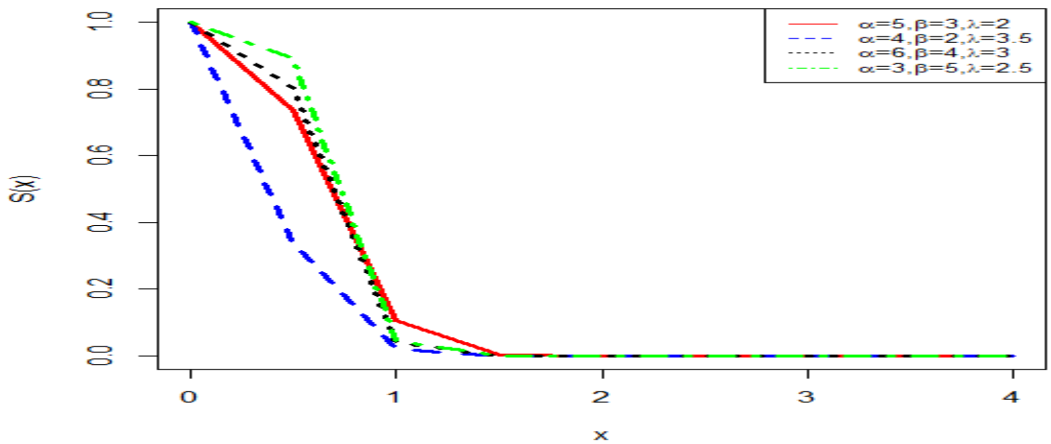

Figure 2: show the shapes of the SF of the MTW distribution

4. Hazard Rate Function (HRF)

The expression for hazard rate function or failure rate is expressed as

$$h(x)=\frac{f(x)}{S(x)} \qquad (11)$$

Inserting equation (9) and (10) in (11), we have

$$h(y)=\frac{\dfrac{\beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}(\alpha-2+2e^{-\lambda x^{\beta}})}{\alpha-1}}{1-\dfrac{(1-e^{-\lambda x^{\beta}})(\alpha-1+e^{-\lambda x^{\beta}})}{\alpha-1}}$$

After simplification, the final result for HRF become as under

$$h(x)=\frac{\beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}(\alpha-2+2e^{-\lambda x^{\beta}})}{\alpha-1-\left((1-e^{-\lambda x^{\beta}})(\alpha-1+e^{-\lambda x^{\beta}})\right)} \qquad (12)$$

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5. The Reversed Hazard Rate Function (RHRF)

The expression for reversed hazard rate function is

$$r(x)=\frac{f(x)}{F(x)}$$

$$r(x)=\frac{\dfrac{\beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}(\alpha-2+2e^{-\lambda x^{\beta}})}{\alpha-1}}{\dfrac{(1-e^{-\lambda x^{\beta}})(\alpha-1+e^{-\lambda x^{\beta}})}{\alpha-1}}$$

$$r(x)=\frac{\beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}(\alpha-2+2e^{-\lambda x^{\beta}})}{\left((1-e^{-\lambda x^{\beta}})(\alpha-1+e^{-\lambda x^{\beta}})\right)} \qquad (13)$$

6. Quantile function (QF)

Quantiles are valuable characteristics because they are less affected by extreme values. Let X denotes a r.v with MTW distribution, then the quantile function of the MTW distribution is given as

$$F(x)=u$$

using equation (8), we get

$$\frac{\left(1-e^{-\lambda x^{\beta}}\right)\left(\alpha-1+e^{-\lambda x^{\beta}}\right)}{\alpha-1}=u$$

$$\left(e^{-\lambda x^{\beta}}\right)^{2}-\left(2-\alpha\right)e^{-\lambda x^{\beta}}+u\left(\alpha-1\right)+\left(\alpha-1\right)=0 \qquad (14)$$

If we put $y=e^{-\lambda x^{\beta}}$ in equation (14), we get the quadratic equation form

$$y^{2}+\left(2-\alpha\right)y+u\left(\alpha-1\right)+\left(\alpha-1\right)=0 \qquad (15)$$

According to quadratic formula

$$y=\frac{-b\pm\sqrt{b^{2}-4ac}}{2a} \qquad (16)$$

Here, from equation (15)

$a=1 \qquad b=2-\alpha \qquad c=u(\alpha-1)+\left(2-\alpha\right)$

Putting the values of a, b, c in equation (16), after simplification we obtained

$$y = \frac{(2-\alpha)+\sqrt{\alpha^2-4\alpha u+4u}}{2} \tag{17}$$

Replacing y by $e^{-\lambda x^{\beta}}$ in equation (17), we get as

$$e^{-\lambda x^{\beta}} = \frac{(2-\alpha)+\sqrt{\alpha^2-4\alpha u+4u}}{2} \tag{18}$$

Solving equation (18), we obtain

$$x = \left(\frac{-1}{\lambda}\log\left(\frac{(2-\alpha)+\sqrt{\alpha^2-4\alpha u+4u}}{2}\right)\right)^{\frac{1}{\beta}} \tag{19}$$

Lemma.1 The median of the MTW distribution can be obtained if $q = \frac{1}{2}$

$$x = \left(\frac{-1}{\lambda}\log\left(\frac{(2-\alpha)+\sqrt{\alpha^2-2\alpha-2}}{2}\right)\right)^{\frac{1}{\beta}} \tag{20}$$

Lemma 2: if $U = \frac{1}{8}$, then we obtained the octile of the MTW distribution.

$$x = \left(\frac{-1}{\lambda}\log\left(\frac{(2-\alpha)+\sqrt{\alpha^2-\frac{\alpha}{2}-\frac{1}{2}}}{2}\right)\right)^{\frac{1}{\beta}}$$

$$x = \left(\frac{-1}{\lambda}\log\left(\frac{(2-\alpha)+\sqrt{\frac{\alpha(2\alpha-1)+1}{2}}}{2}\right)\right)^{\frac{-1}{\beta}} \tag{21}$$

Lemma 3: The decile function of the MTW distribution can be obtained by putting $U = \frac{1}{10}$

$$x = \left(\frac{-1}{\lambda}\log\left(\frac{(2-\alpha)+\sqrt{\alpha^2-\frac{2\alpha}{5}-\frac{2}{5}}}{2}\right)\right)^{\frac{-1}{\beta}}$$

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$$x = \left(\frac{-1}{\lambda} \log \left(\frac{(2-\alpha) + \sqrt{\frac{5\alpha^2 - 2(\alpha-1)}{5}}}{2} \right) \right)^{\frac{-1}{\beta}} \qquad (23)$$

The other quantiles order can be found by putting different values of U

7. The r^{th} moments

Let x be a r.v, which follows the modified transmuted Weibull (MTW) distribution, then the r^{th} moments about the origin as

$$E(x^r) = \mu_r^{/} = \int_0^\infty x^r f(x) dx \qquad (24)$$

Using equation (9), equation (24) becomes

$$E(x^r) = \int_0^\infty \frac{x^r \beta \lambda x^{\beta-1} e^{-\lambda x^\beta} \left(\alpha - 2 + 2e^{-\lambda x^\beta} \right) dx}{\alpha - 1} \qquad (25)$$

Let we suppose that

$$y = x^\beta \qquad dy = \beta x^{\beta-1} dx \qquad x = y^{\frac{1}{\beta}} \qquad 0 < x < \infty,\ 0 < y < \infty$$

Equation (25) takes the form

$$E(x^r) = \frac{1}{\alpha - 1} \int_0^\infty y^{\frac{r}{\beta}} \lambda e^{-\lambda y} \left(\alpha - 2 + 2e^{-\lambda y} \right) dy \qquad (26)$$

Again, we suppose that

$$z = e^{-\lambda y}, \quad \log z = -\lambda y, \quad y = -\frac{-1}{\lambda} \log z, \quad -dz = \lambda e^{-\lambda y} dy, \quad 0 < y < \infty, 1 < z < 0,$$

$$m = \frac{r}{\beta}$$

$$E(x^r) = \frac{1}{\alpha - 1} \int_1^0 \left(\frac{-1}{\lambda} \log z \right)^m \left(\alpha - 2 + 2z \right)(-dz)$$

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Applying transformation, we have

$$E(x^r)=\frac{\left(\frac{-1}{\lambda}\right)^m}{(\alpha-1)}\left((\alpha-2)\int_0^1(\log z)^m dz+2\int_0^1(\log z)^m zdz\right) \tag{27}$$

Let $\log z=u$ $\quad \frac{1}{z}dz=du$ $\quad dz=zdu$ $\quad 0<z<1$ $\quad -\infty<u<0$

$z=e^u$ $\quad \log z=u$ $\quad \frac{1}{z}dz=du$ $\quad dz=e^u du$

Again, using transformation, we have

$$\mu_r^/=\frac{\left(\frac{-1}{\lambda}\right)^m}{\alpha-1}\left((\alpha-2)\int_{-\infty}^0 u^m e^u du+2\int_{-\infty}^0 u^m e^m e^m du\right)$$

$$\mu_r^/=\frac{\left(\frac{-1}{\lambda}\right)^m}{\alpha-1}\left((\alpha-2)\int_0^\infty u^m e^u du+2\int_0^\infty u^m e^{2m} du\right) \tag{28}$$

Using the identity

$$\int_0^\infty u^m e^{u(k+1)}du=\frac{(k+1)^{-m-1}\overline{|m+1,-(k+1)u}}{(-1)^m}$$

Now, we have the integral parts from equation (28) is

$$\int_0^\infty u^m e^u du=\frac{\overline{|m+1,-u}}{(-1)^m}$$

$$\int_0^\infty u^m e^{2u} du=\frac{\overline{|m+1,-2u}}{(-1)^m}$$

Equation (28) takes the form

$$\mu_r^{\,/}=\frac{(-1)2^{\frac{r}{\beta}}(\alpha-2)\overline{\left|\left(\frac{r+\beta}{\beta},\lambda x^\beta\right)\right.}+\overline{\left|\left(\frac{r+\beta}{\beta},2\lambda x^\beta\right)\right.}}{2^{\frac{r}{\beta}}(\alpha-1)\lambda^{\frac{r}{\beta}}} \tag{29}$$

From equation (29), we can easily obtain mean, variance, skewness and kurtosis.

8. Maximum Likelihood Estimates

To find parameters of the MTW distribution, we consider the usual MLEs procedure to estimate the involved parameters. The likelihood function is defined as

$$L=\prod_{i=0}^{\infty} f(x,\alpha,\beta,\lambda)$$

$$L=\prod_{i=0}^{\infty}\left(\frac{\beta\lambda x^{\beta-1}e^{=\lambda x^{\beta}}\left(\alpha-2+2e^{=\lambda x^{\beta}}\right)}{\alpha-1}\right)$$

$$\log L=\sum_{i=1}^{n}\left(\log\beta+\log\lambda+(\beta-1)\log x-\lambda x^{\beta}+\log(\alpha-2+2e^{-\lambda x^{\beta}})-\log(\alpha-1)\right) \qquad (30)$$

Differentiating equation (30) w.r.t α , β , λ and equating to zero, we have the following results

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\sum_{i=1}^{n}\left(\alpha-2+2e^{-\lambda x^{\beta}}\right)}-\frac{n}{\alpha-1}=0$$

$$\frac{\partial\log L}{\partial\alpha}=n(\alpha-1)-n\sum_{i=1}^{n}(\alpha-2+2e^{-\lambda x^{\beta}})=0 \qquad (31)$$

$$\frac{\partial\log L}{\partial\beta}=\sum_{i=1}^{n}\left(\frac{1}{\beta}+\log x-\lambda x^{\beta}\log x-\frac{2\lambda x^{\beta}e^{-\lambda x^{\beta}}\log x}{\left(\alpha-2+2e^{-\lambda x^{\beta}}\right)}\right)=0$$

$$\frac{\partial\log L}{\partial\beta}=\frac{n}{\beta}+\sum_{i=0}^{n}\log x-\lambda\sum_{i=0}^{n}x^{\beta}\log x-2\lambda\sum_{i=0}^{n}\left(\frac{x^{\beta}e^{-\lambda x^{\beta}}\log x}{\alpha-2+2e^{-\lambda x^{\beta}}}\right)=0 \qquad (32)$$

$$\frac{\partial\log L}{\partial\lambda}=\sum_{i=1}^{n}\left(\frac{1}{\lambda}-x^{\beta}+\frac{2(-x^{\beta})e^{-\lambda x^{\beta}}}{\left(\alpha-2+2e^{-\lambda x^{\beta}}\right)}\right)=0$$

$$\frac{\partial\log L}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=0}^{n}x^{\beta}-2\sum_{i=0}^{n}\left(\frac{x^{\beta}e^{-\lambda x^{\beta}}}{\alpha-2+2e^{-\lambda x^{\beta}}}\right)=0 \qquad (33)$$

The MLEs can be found by the solution of equations (31), (32) and (33) through computer software

9. Order Statistics

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Let the ordered random variables of size n from the MTW distribution are $x_1, x_2, x_2, ...x_n$, then the general form of the PDF of ordered statistic is

$$f_{i:n}(x) = \frac{n!}{(n-i)!(i-1)!}\left(F(x)\right)^{i-1} f(x)\left(1-F(x)\right)^{n-i} \qquad (34)$$

Putting PDF and CDF of the MTW distribution, equation (34) become as

$$f_{i:n}(x) = \frac{n!}{(n-i)!(i-1)!}\left(\frac{\left(1-e^{-\lambda x^{\beta}}\right)\left(\alpha-1+e^{-\lambda x^{\beta}}\right)}{(\alpha-1)}\right)^{i-1}\left(\frac{\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}\left(\alpha-2+2e^{-\lambda x^{\beta}}\right)}{(\alpha-1)}\right)\left(\frac{\left(1-e^{-\lambda x^{\beta}}\right)\left(\alpha-1+e^{-\lambda x^{\beta}}\right)}{(\alpha-1)}\right)^{n-i}$$

$$f_{i;n} = \frac{\beta\lambda n!}{(\alpha-1)^n (n-i)!(i-1)!}\left((1-e^{-\lambda x^{\beta}})(\alpha-1+e^{-\lambda x^{\beta}})\right)^{i-1} \left(x^{\beta-1}e^{-\lambda x^{\beta}}(\alpha-2+2e^{-\lambda x^{\beta}})\right)\left(1-(1-e^{-\lambda x^{\beta}})(\alpha-1+e^{-\lambda x^{\beta}})\right)^{n-i} \qquad (35)$$

The first order statistic can be written as

$$f_1(x) = \frac{\beta\lambda n}{(\alpha-1)^n}\left(x^{\beta-1}e^{-\lambda x^{\beta}}(\alpha-2+2e^{-\lambda x^{\beta}})\right)\left((\alpha-2+e^{-\lambda x^{\beta}})(\alpha-1+e^{-\lambda x^{\beta}})\right)^{n-1} \qquad (36)$$

The n^{th} order statistic can be written as

$$f_n(x) = \frac{\beta\lambda n}{(\alpha-1)^n}\left(x^{\beta-1}e^{-\lambda x^{\beta}}(\alpha-2+2e^{-\lambda x^{\beta}})\right)\left((1-e^{-\lambda x^{\beta}})(\alpha-1+e^{-\lambda x^{\beta}})\right)^{n-1} \qquad (37)$$

10. Mode of the MTW Distribution

If "X" is a r.v having distribution function as MTW distribution, then the mode is defined as

$$\frac{d}{dx}\left(f(x)\right) = 0 \qquad (38)$$

Inserting the PDF of MTW distribution in equation (38), we have

$$\frac{d}{dx}\left(\frac{\beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}\left(\alpha-2+2e^{-\lambda x^{\beta}}\right)}{\alpha-1}\right) = 0 \qquad (39)$$

$$-\alpha\lambda x^{2\beta-1}e^{\lambda x^{\beta}} + 2\lambda\beta x^{2\beta-2}e^{\lambda x^{\beta}} - 4\lambda\beta x^{2\beta-2} + \alpha(\beta-1)x^{\beta-2}e^{\lambda x^{\beta}} - 2(\beta-1)x^{\beta-2}e^{\lambda x^{\beta}} + 2(\beta-1)x^{\beta-2} = 0$$

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After simplification, we have the expression as

$$e^{\lambda x^{\beta}}\left(2\lambda\beta x^{\beta}-\alpha\lambda x^{\beta}+\alpha(\beta-1)\right)-4\lambda\beta x^{\beta}+2(\beta-1)=0 \qquad (40)$$

Using statistical software, we can find the solution of (40).

11. Simulation study

Simulation study has been carried out for obtaining average Bias and MSE. The simulated data were generated 1000 times for different sample size. The random number were generated by the following expression

$$x=\left(\frac{-1}{\lambda}\log\left(\frac{(2-\alpha)+\sqrt{\alpha^{2}-4\alpha u+4u}}{2}\right)\right)^{\frac{1}{\beta}}$$

Average bias and mean square error (MSE) were calculated by the following expression

$$Bias=\frac{1}{w}\sum_{i=}^{w}(\hat{\beta}_i-\beta)$$

$$MSE=\frac{1}{w}\sum_{i=0}^{w}(\hat{\beta}_i-\beta)^2$$

Table 1: Bias and MSE for the parameters α, β, λ

α	β	λ	n	MSE(α)	MSE(β)	MSE(λ)	Bias(α)	Bias(β)	Bias(λ)
5.0	4.0	3.5	30	4.5976	0.4403	1.5936	0.3621	0.2133	0.3488
			50	4.0914	0.2558	0.7570	0.1578	0.1342	0.1373
			70	2.3692	0.1682	0.5019	0.1575	0.0960	0.0814
			80	1.3293	0.1479	0.4033	0.1490	0.0799	0.0775
			90	1.1766	0.1227	0.3234	0.1413	0.0755	0.0525
4.5	4.0	3.5	30	4.7425	0.4308	1.7052	0.6457	0.2018	0.3898
			50	4.1515	0.2521	0.7978	0.4350	0.1227	0.1652
			70	2.2275	0.1639	0.5354	0.0909	0.0904	0.0854
			80	1.3532	0.1463	0.4289	0.0907	0.0778	0.0538
			90	1.2275	0.1220	0.3471	0.1223	0.0734	0.0288

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4.5	4.5	3.5	30	4.8428	0.5501	1.6784	0.6652	0.2274	0.3871
			40	4.3757	0.3889	1.1241	0.4600	0.1647	0.2356
			70	4.2685	0.2062	0.5793	0.4165	0.0961	0.0654
			80	2.7768	0.1824	0.4520	0.1618	0.0823	0.0410
			90	1.7020	0.1518	0.3603	0.0097	0.0799	0.0332
5.5	4.5	3.5	30	4.4256	0.5672	1.5187	0.1948	0.2501	0.3302
			40	4.3929	0.4063	1.0052	0.1415	0.1861	0.1920
			50	3.6965	0.3271	0.7246	0.1329	0.1591	0.1313
			80	3.0276	0.1932	0.3964	0.1301	0.0960	0.0355
			90	1.5905	0.1574	0.3027	0.1245	0.0834	0.0232

Table 1 presents the average bias and the MSE of the estimates. The values of the bias and the MSEs decrease when n increases.

12. Application

We consider a real data set to prove the importance and flexibility of the modified transmuted Weibull distribution. This data set have fitted to modified transmuted Weibull distribution and compared with some extension of Weibull distribution. Some goodness of fit and p values has been produced. The model with minimum goodness of fit is said to be the best fit to the data.

Different goodness of fits are defined by

$$AIC = -2l(\hat{\vartheta}) + \Psi$$

$$BIC = -2l(\hat{\vartheta}) + \Psi \log(n)$$

$$HQIC = -2l(\hat{\vartheta}) + 2\Psi \log(\log(n))$$

$$CAIC = -2l(\hat{\vartheta}) + \frac{2\Psi n}{n - \Psi - 1}$$

Data set: remission time (in month) of bladder cancer Patients

The data set corresponding to remission time (in month) given by Lee and Wang (2003) and consists of 128 random sample of bladder cancer patients. The data has been modelled using different distributions in Table 2.

Table 2: MLEs and p- values for bladder cancer data

Distribution	MLEs			Statistics	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	-LogL	p-value

MT Weibull	2.3003	1.1269	0.0488	411.9923	0.6827
Weibull	1.0490	0.0939	_	414.088	0.5763
NGIW	-0.4670	0.4161	5.8888/10.0337	413.8239	0.5575
NAPTW	5.5093	0.7531	0.3452	412.4323	0.7035
IW	3.2573	0.7522	_	444.0008	0.0125

Table 3: Measure of AIC, BIC, HQIC, and CAIC for bladder cancer data

Distribution	AIC	CAIC	BIC	HQIC
MT Weibull	**829.8146**	**830.1781**	**838.5407**	**833.461**
Weibull	832.1766	832.2626	837.8806	834.4942
NGIW	835.6477	835.9729	847.0559	840.2829
NAPTW	830.8645	831.0581	839.4206	834.3409
IW	892.0015	892.0975	897.7056	894.3191

Table 3 describes the values of AIC, CAIC, BIC and HQIC. The results reveal that the modified transmuted Weibull distribution provides better fit than the other version of Weibull Distribution.

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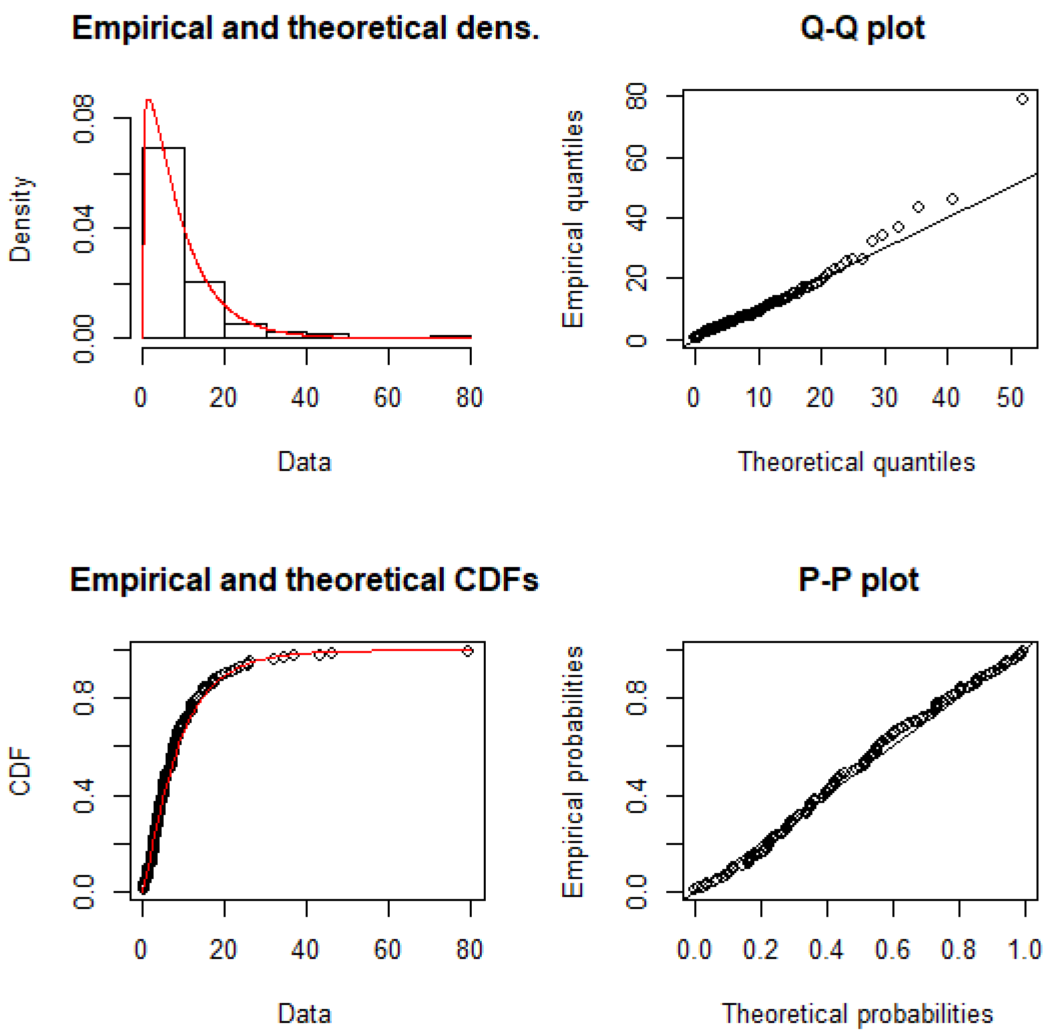

Figure 3 Theoretical and empirical PDF and CDF with Q-Q and P-P plot of the MTW distribution

Figure 3 illustrates the theoretical and empirical PDF, CDF, Q-Q and P-P plot of the MTW distribution using the bladder cancer patient's data and is observed that the MTW distribution is the best fitted line as compared to other distributions. The TTT plot has been given in Figure 4.

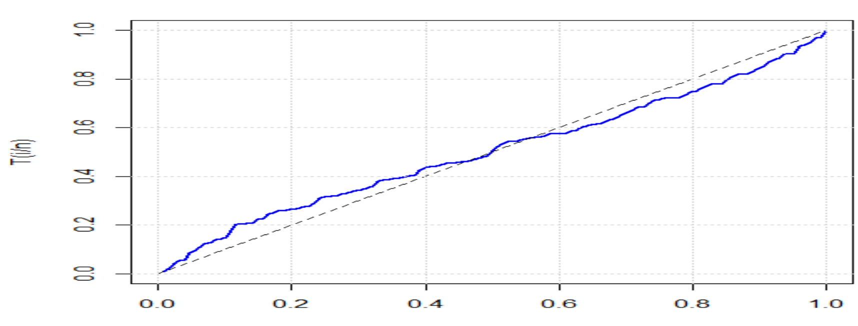

Fig 4: TTT plot for the data of bladder cancer

The TTT plot in fig 4 clearly shows that this data follows a non-monotonic hazard rate shape.

13. Concluding Remarks

Several distributions have been discussed in section 2 to the problem of determining which of a model best describes the lifetime (bladder cancer patient's) data. Different goodness of fits (AIC, BIC, CAIC, HQIC) of various model using bladder cancer data are presented here. If we see the values of AIC, BIC, CAIC, HQIC for the proposed model, it gives smaller values than others models. Hence, we concluded that modified transmuted Weibull distribution leads to a better. Based on the findings we concluded that the new constructed model gave better result and may be applied to a real phenomenon.

6. List of Abbreviations

MTW	Modified Transmuted Weibull
MT	Modified Transmuted
CDF	Cumulative Distribution Function
APT	Alpha Power Transformation
PDF	Probability density function
SF	Survival Function
HRF	Hazard Rate Function
RHRF	Reversed Hazard Rate Function
IQR	Inter Quartile Range
MGF	Moment Generating Function
MLEs	Maximum Likelihood Estimates
MSE	Mean Square Error
AIC	Akaike Information Criterion
BIC	Bayesian information criterion
CAIC	Consistent Akaike Information Criterion
HQIC	Hannan and Quinn Information Criterion
NAPTW	New Alpha Power Transform Weibull
MRF	Mean Residual Function
MRL	Mean Residual Life
MWT	Mean Waiting Time

Availability of data and materials

All the relevant data and materials are available and are cited in the manuscript.

Competing Interest

Not Applicable

Funding

There is no funding body, we are going to pay the Article Processing Charges (APC) from our pockets.

Acknowledgments

I would appreciate the valuable comments and further suggestions of the referees for the improvement of the article

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